

Test functions and wavefront sets at positive depth

Joint work with Dan Ciubotaru

Wavefront Sets of Representations and Endoscopic Classification

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Local character expansion: notation

Let $F/\mathbb{Q}_p$ be finite, with residue field $\mathbb{F}_q$, and let $\mathbf{G}$ be connected reductive. Write $G = \mathbf{G}(F)$, $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})(F)$ and $(\pi, V) \in \mathrm{Irr}(G)$. Let Θ_π be the character distribution of π .

Write $U_\pi \subset G$ for the domain of validity of the local character expansion. For $\mathrm{supp}(f) \subset U_\pi$,

$$\Theta_\pi(f) = \sum_{\mathcal{O} \in \mathcal{N}(\mathfrak{g}^*)/G} c_{\mathcal{O}}(\pi) \hat{\mu}_{\mathcal{O}}(f \circ \exp).$$

Here $\mu_{\mathcal{O}}$ is a nilpotent orbital integral and $\hat{\mu}_{\mathcal{O}}$ its Fourier transform.

$$\mathrm{WF}(\pi) = \max\{\mathcal{O} : c_{\mathcal{O}}(\pi) \neq 0\}.$$

For $\mathrm{WF}(\pi)$, maximality uses closure in the p -adic topology. For $\mathrm{WF}_{\mathrm{geom}}(\pi)$, pass to geometric orbits and take the maximal ones with respect to Zariski closure.

Motivation: Barbasch–Moy test functions

The Schneider–Stuhler equivariant coefficient system associated to a depth-0 representation π assigns

$$\mathcal{H}_\pi : x \in \mathcal{B}(G) \mapsto \pi^{G_{x,0^+}} \quad (1)$$

on the Bruhat–Tits building. Each stalk carries an action of $G_{x,0}$ that factors through the finite group of Lie type $G_{x,0}/G_{x,0^+}$.

In [BM97], Barbasch–Moy constructed

1. a lifting map $\mathcal{L}_{x,0} : \mathcal{N}(\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}) \rightarrow \mathcal{N}(\mathfrak{g})$, that sends $\bar{e}$ to the unique minimal orbit that intersect $\bar{e} + \mathfrak{g}_{x,0^+}$,
2. test functions $f_{x,\bar{e}}^{BM} \in C_c^\infty(G)$ that compute the finite field rational wavefront set of the stalk $\mathcal{H}_{\pi,x}$, with properties that ensure that

$$\mathrm{WF}(\pi) = \cup_x \mathcal{L}_{x,0}(\mathrm{WF}_{\mathbb{F}_q}(\pi^{G_{x,0^+}})). \quad (2)$$

Three properties of a detecting family

Write $\mathcal{O} \leq \mathcal{O}'$ for rational orbit closure. Seek a test $f_{\mathcal{O}}$ for each nilpotent orbit $\mathcal{O}$, with:

1. Support

$$\text{supp}(f_{\mathcal{O}}) \subset U_{\pi}.$$

2a. Triangularity

$$\hat{\mu}_{\mathcal{O}'}(f_{\mathcal{O}} \circ \exp) \neq 0 \implies \mathcal{O} \leq \mathcal{O}'.$$

2b. Nonzero diagonal

$$\hat{\mu}_{\mathcal{O}}(f_{\mathcal{O}} \circ \exp) \neq 0.$$

The tests recover the wavefront set

Recalling that

$$\Theta_{\pi}(f) = \sum_{\mathcal{O} \in \mathcal{N}(\mathfrak{g}^*)/G} c_{\mathcal{O}}(\pi) \hat{\mu}_{\mathcal{O}}(f \circ \exp)$$

it is easy to see that these three properties imply

$$\text{WF}(\pi) = \max\{\mathcal{O} : \Theta_{\pi}(f_{\mathcal{O}}) \neq 0\}.$$

Thus computing the leading orbits reduces to computing character evaluations on a suitable family of tests.

The representation attached to the nilpotent datum

For $x \in \mathcal{B}(G)$, recall $\overline{G}_x(\mathbb{F}_q) = G_{x,0}/G_{x,0+}$ and put $\overline{\mathfrak{g}}_x = \text{Lie}(\overline{G}_x)(\mathbb{F}_q)$.

Choose an $\mathfrak{sl}_2$ -triple $(\bar{e}, \bar{h}, \bar{f})$, with weight spaces $\overline{\mathfrak{g}}_x(j)$, and put

$$\mathfrak{u} = \bigoplus_{j>0} \overline{\mathfrak{g}}_x(j), \quad U = \exp(\mathfrak{u}).$$

Fix an invariant nondegenerate pairing B and a nontrivial additive character ψ of $\mathbb{F}_q$. Define

$$\lambda \in \mathfrak{u}^*, \quad \lambda(Y) = -B(\bar{f}, Y), \quad \Omega = U \cdot \lambda \subset \mathfrak{u}^*.$$

The orbit method gives an irreducible representation $\eta_{\bar{e}}$ of U whose character is

$$\text{tr } \eta_{\bar{e}}(\exp Y) = |\Omega|^{-1/2} \sum_{\ell \in \Omega} \psi(\ell(Y)).$$

Barbasch–Moy tests at depth zero

The generalised Gelfand–Graev representation and its character are

$$\Gamma_{\bar{e}} = \text{Ind}_U^{\overline{G}_x(\mathbb{F}_q)} \eta_{\bar{e}}, \quad \gamma_{\bar{e}} = \text{tr } \Gamma_{\bar{e}}.$$

Inflate to $G_{x,0}$ and extend by zero:

$$f_{x,\bar{e}}^{\text{BM}}(g) = \begin{cases} \gamma_{\bar{e}}(\bar{g}), & g \in G_{x,0}, \\ 0, & g \notin G_{x,0}. \end{cases}$$

For depth-zero π , Waldspurger, and later DeBacker, proved that the local character expansion is valid on

$$G_{0+} := \bigcup_{y \in \mathcal{B}(G)} G_{y,0+}.$$

Thus property 1 reduces to showing $\text{supp } f_{x,\bar{e}}^{\text{BM}} \subset G_{0+}$.

Support and character evaluations

The character $\gamma_{\bar{e}}$ is supported on unipotent elements. Moy–Prasad proved that if $\bar{g} \in \overline{G}_x(\mathbb{F}_q)$ is unipotent then $\bar{g}G_{x,0+} \subset G_{y,0+}$ for some y and so

$$\text{supp } f_{x,\bar{e}}^{\text{BM}} \subset G_{0+} \subset U_{\pi} \quad (\text{depth}(\pi) = 0).$$

Writing $\theta_{\pi,x}$ for the character on $V^{G_{x,0+}}$, inflation gives

$$\Theta_{\pi}(f_{x,\bar{e}}^{\text{BM}}) = \frac{\text{vol}(G_{x,0})}{|\overline{G}_x(\mathbb{F}_q)|} \sum_{h \in \overline{G}_x(\mathbb{F}_q)} \theta_{\pi,x}(h) \gamma_{\bar{e}}(h).$$

Thus the evaluations are the finite-group character pairings that define the finite-field wavefront set.

The advantage of reduction to finite fields

The reduction to finite fields is advantageous because the representation theory of finite reductive groups is much more completely understood.

Mœglin–Waldspurger approach the same invariant through generalised Whittaker models of the p -adic representation. Generalised Gelfand–Graev representations supply their finite-field counterparts. [MW87]

Depth-zero supercuspidals: a finite-group formula

Let $\pi = \pi(x, \sigma)$ be depth-zero supercuspidal, with σ cuspidal on $\overline{G}_x(\mathbb{F}_q)$ and character θ_σ .

Ciubotaru–Mason–Brown–O. [CMO24]

$$\mathrm{WF}(\pi) = \max \left\{ \mathcal{L}_{x,0}(\bar{e}) : \langle \overline{\theta_\sigma}, \gamma_{\bar{e}} \rangle_{\overline{G}_x(\mathbb{F}_q)} \neq 0 \right\}.$$

Here $\bar{e}$ ranges over rational nilpotent quotient orbits, $\gamma_{\bar{e}}$ is the generalised Gelfand–Graev character, and $\mathcal{L}_{x,0}$ is orbit lifting.

In particular, $\mathrm{WF}_{\mathrm{geom}}(\pi)$ is a singleton, readily computable using Lusztig's finite-group theory.

Langlands parameters: results and the upper bound

Let $\mathcal{O}_\pi^\vee$ be the nilpotent parameter orbit, $d^\vee$ Barbasch–Vogan duality, and $D\pi = |\mathrm{AZ}(\pi)|$.

- **Ciubotaru–Mason–Brown–O.** [CMO24] For unipotent supercuspidals of semisimple groups inner to split,

$$\mathrm{WF}_{\mathrm{geom}}(\pi) = \{d^\vee(\mathcal{O}_\pi^\vee)\}.$$

- **Ciubotaru–Kim.** [CK24], Theorem 1.8. For depth-zero supercuspidals of symplectic, special orthogonal and unitary groups,

$$\mathrm{WF}_{\mathrm{geom}}(\pi) \leq d^\vee(\mathcal{O}_\pi^\vee).$$

- **Upper bound conjecture: Ciubotaru–Kim; Hazeltine–Liu–Lo–Shahidi.**

$$\mathrm{WF}_{\mathrm{geom}}(D\pi) \leq d^\vee(\mathcal{O}_\pi^\vee).$$

Test functions for positive depth

The algebraic structure at positive depth

What algebraic structures are there at (rational) positive depth?

Moy–Prasad filtrations and finite quotients

Let $s \in \mathbb{R}$ and $x \in \mathcal{B}(G)$. Moy–Prasad give filtrations $G_{x,s}$ and $\mathfrak{g}_{x,s}$, with

$$G_{x,s^+} = \bigcup_{t>s} G_{x,t}, \quad \mathfrak{g}_{x,s^+} = \bigcup_{t>s} \mathfrak{g}_{x,t}.$$

At depth zero we have the reductive quotient

$$\overline{G}_x(\mathbb{F}_q) = G_{x,0}/G_{x,0^+}, \quad \mathrm{Lie}(\overline{G}_x)(\mathbb{F}_q) = \underline{\mathfrak{g}}_{x,0}.$$

For $s > 0$, the (mock) exponential map identifies:

$$G_{x,s}/G_{x,s^+} \simeq \underline{\mathfrak{g}}_{x,s} := \mathfrak{g}_{x,s}/\mathfrak{g}_{x,s^+}.$$

Conjugation induces an action of $\overline{G}_x(\mathbb{F}_q)$ on $\underline{\mathfrak{g}}_{x,s}$.

The filtration produces a graded Lie algebra

The Lie bracket respects depth:

$$[\mathfrak{g}_{x,s}, \mathfrak{g}_{x,t}] \subseteq \mathfrak{g}_{x,s+t}.$$

Hence it descends to

$$[\ , \] : \underline{\mathfrak{g}}_{x,s} \times \underline{\mathfrak{g}}_{x,t} \longrightarrow \underline{\mathfrak{g}}_{x,s+t}.$$

Taking depths modulo $\mathbb{Z}$ gives the $\mathbb{R}/\mathbb{Z}$ -graded Lie algebra

$$\bigoplus_{s \in \mathbb{R}/\mathbb{Z}} \underline{\mathfrak{g}}_{x,s}.$$

Rational points give cyclic gradings

At a rational point $x \in \mathcal{B}(G)$ ($a(x) \in \mathbb{Q}$ for all affine a), the $\mathbb{R}/\mathbb{Z}$ -grading simplifies to a $\mathbb{Z}/n$ -grading: $\langle r \in \mathbb{R}/\mathbb{Z} : \underline{\mathfrak{g}}_{x,r} \neq 0 \rangle \simeq \mathbb{Z}/n$ with generator $r_0 \in \mathbb{Q}$,

$$\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}/n} \mathfrak{l}_i, \quad \mathfrak{l}_i = \underline{\mathfrak{g}}_{x, ir_0}, \quad [\mathfrak{l}_i, \mathfrak{l}_j] \subseteq \mathfrak{l}_{i+j}.$$

If $r = ir_0$, the action we need is

$$\overline{G}_x \curvearrowright \mathfrak{l}_i = \underline{\mathfrak{g}}_{x,r}, \quad \text{Lie}(\overline{G}_x) = \mathfrak{l}_0.$$

We can simplify further to $i = 1$ by restricting our attention to $\mathfrak{l}' = \bigoplus_j \mathfrak{l}_{ij}$.

At $r \in \mathbb{Z}$, this gives $n = 1$: this is an ordinary reductive Lie algebra with its adjoint action.

Example: the Siegel facet of Sp_{2n}

Take $G = \mathrm{Sp}_{2n}(\mathbb{Q}_p)$, and the standard alcove

$$\frac{1}{2} \geq x_1 \geq \cdots \geq x_n \geq 0.$$

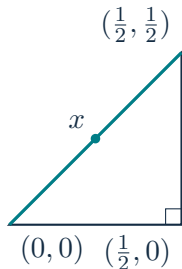
Choose $x = (\frac{1}{4}, \dots, \frac{1}{4})$ on the Siegel edge and set $r = \frac{1}{2}$.

The root values are

$$\begin{aligned}(\varepsilon_i - \varepsilon_j)(x) &= 0, \\(\varepsilon_i + \varepsilon_j)(x) &= \frac{1}{2}, \\(2\varepsilon_i)(x) &= \frac{1}{2}.\end{aligned}$$

Thus only degrees 0 and $\frac{1}{2}$ occur modulo $\mathbb{Z}$, and $\overline{G}_x \simeq \mathrm{GL}_n$.

Test functions for positive depth



The Sp_4 alcove; the Siegel edge is highlighted.

The Siegel $\mathbb{Z}/2$ -grading in block matrices

With $W = \mathbb{F}_q^n$, write

$$\mathfrak{sp}_{2n}(\mathbb{F}_q) = \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} : B = B^\top, C = C^\top \right\}.$$

At $x = (\frac{1}{4}, \dots, \frac{1}{4})$ and $r = \frac{1}{2}$, we obtain

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^\top \end{pmatrix} \right\} \simeq \mathfrak{gl}(W),$$
$$\mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} : B = B^\top, C = C^\top \right\} \simeq \mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^*.$$

The reductive quotient $\mathrm{GL}(W)$ acts by

$$g \cdot (B, C) = (gBg^\top, g^{-\top}Cg^{-1}).$$

This is the Vinberg representation for $\vartheta = \mathrm{Ad}(\mathrm{diag}(I_n, -I_n))$ on Sp_{2n} .

From π to a representation of a finite vector space

Focus on the data $(\overline{G}_x, \underline{\mathfrak{g}}_{x,r})$. Taking fixed vectors gives

$$V_{x,r} := V^{G_{x,r}+}, \quad \pi_{x,r} : \underline{G}_{x,r} \longrightarrow \mathrm{GL}(V_{x,r}),$$

where $\underline{G}_{x,r} = G_{x,r}/G_{x,r}+$.

Via $\exp : \underline{\mathfrak{g}}_{x,r} \xrightarrow{\sim} \underline{G}_{x,r}$, this is a representation of the additive group $\underline{\mathfrak{g}}_{x,r}$; write $\theta_{\pi,x,r}$ for its character.

The additive characters are $Y \mapsto \psi(B(X, Y))$, with $X \in \underline{\mathfrak{g}}_{x,-r}$. Equivariance makes their multiplicities constant on $\overline{G}_x(\mathbb{F}_q)$ -orbits:

$$\theta_{\pi,x,r}(Y) = \sum_{\mathcal{O} \in \overline{G}_x(\mathbb{F}_q) \backslash \underline{\mathfrak{g}}_{x,-r}} m_{\pi,x,r}(\mathcal{O}) \sum_{X \in \mathcal{O}} \psi(B(X, Y)).$$

Here $m_{\pi,x,r}(\mathcal{O})$ is the common multiplicity of the characters indexed by $\mathcal{O}$.

Construction of the test functions

Why work at the depth of the representation?

Recall: Schneider–Stuhler theory and its positive-depth refinements recover π from its depth- r fixed-vector data and equivariant gluing, with $r = \text{depth}(\pi)$. Hence its wavefront set should also be recoverable from that data. [SS97]

Practical considerations: At $r = \text{depth}(\pi)$, the spaces $V^{G_{x,r+}}$ often have a simple structure that we can control directly (with the type). At greater depth, these spaces grow and their structure becomes harder to determine.

Detecting families at depth r

We seek positive-depth analogues of the Barbasch–Moy test functions satisfying conditions 1, 2a and 2b below, for each rational nilpotent orbit $\mathcal{O}$.

For $\text{depth}(\pi) = r > 0$, DeBacker's theorem lets us take $U_\pi = G_{r+}$. [DH02]

1. Support

$$\text{supp}(f) \subset G_{r+} := \bigcup_y G_{y,r+}.$$

2a. Triangularity

$$\hat{\mu}_{\mathcal{O}'}(f \circ \exp) \neq 0 \implies \mathcal{O} \leq \mathcal{O}'.$$

2b. Nonzero diagonal

$$\hat{\mu}_{\mathcal{O}}(f \circ \exp) \neq 0.$$

Two sides of the support problem

Fix $r = \text{depth}(\pi)$ and a target orbit $\mathcal{O}$. Let $\phi = f \circ \exp$ be inflated from $\underline{\mathfrak{g}}_{x,s}$.

Support of ϕ	Support of $\hat{\phi}$
Either $s > r$, or $s = r$ and the finite-quotient support is nilpotent .	Must avoid nilpotent orbits $\mathcal{O}'$ unless $\mathcal{O} \leq \mathcal{O}'$.
Ensures the lifted test is supported in $G_{r+} \subset U_\pi$.	Ensures triangularity , since $\hat{\mu}_{\mathcal{O}'}(\phi) = \mu_{\mathcal{O}'}(\hat{\phi})$.

Concentrating Fourier support can spread the original support across the whole quotient.
We must satisfy both conditions at once.

DeBacker optimises Fourier support

Take a coset $C = e + \mathfrak{g}_{x,(-s)+}$ meeting the nilpotent cone, and let $\mathcal{O}$ be its least nilpotent orbit. Set

$$\hat{\phi} = \mathbf{1}_C, \quad \phi = c \psi(-B(e, \bullet)) \mathbf{1}_{\mathfrak{g}_{x,s}}, \quad c > 0.$$

- **Gain:** the small Fourier support C meets only nilpotent orbits above $\mathcal{O}$.
Triangularity is immediate. [D02]
- **Cost:** $\text{supp}(\phi) = \mathfrak{g}_{x,s}$, so the finite-quotient support is the whole vector space, rather than the nilpotent locus.

To guarantee support in G_{r+} , we must take $s > r$. Working at $s = r$ requires a different balance between the two supports.

Enlarge Fourier support to shrink the test support

Choose a nilpotent $e \in \underline{\mathfrak{g}}_{x,-r}$ and complete e to a compatible rational triple (e, h, f) .

Define the **weak slice**

$$\Sigma_e = e + \underline{\mathfrak{g}}_{x,-r}(\leq 0),$$

where (j) denotes the integer weights of the triple.

It has properties similar to the Slodowy slice: it is transverse to the $\overline{G}_x$ -orbit at e , and its slice condition, together with orbit lifting, excludes nilpotent orbits below the target.

Why enlarge the slice? Its affine weight-space description makes the inverse Fourier transform explicit.

The inverse transform has nilpotent support

For the weak slice $\Sigma_e = e + \underline{\mathfrak{g}}_{x,-r}(\leq 0)$, character orthogonality gives

$$\mathcal{F}_{x,r}^{-1}(\mathbf{1}_{\Sigma_e})(Y) = c_e \psi(-B(e, Y)) \mathbf{1}_{\underline{\mathfrak{g}}_{x,r}(\leq -1)}(Y), \quad c_e > 0.$$

Every element of $\underline{\mathfrak{g}}_{x,r}(\leq -1)$ is nilpotent: the inverse cocharacter contracts it to 0.

Both support requirements

The weak slice controls Fourier orbital integrals; its inverse transform is supported on the nilpotent locus of $\underline{\mathfrak{g}}_{x,r}$.

Inflate the finite-field functions to G

Choose $e \in \underline{\mathcal{O}} \subseteq \underline{\mathfrak{g}}_{x,-r}$ and put $H = \overline{G}_x(\mathbb{F}_q)$. The **graded Gelfand–Graev function** is, up to positive normalization, the group average

$$\Gamma_{\underline{\mathcal{O}}}(Y) = c \sum_{h \in H} \mathcal{F}_{x,r}^{-1}(\mathbf{1}_{\Sigma_e})(h \cdot Y), \quad c > 0.$$

Via $\underline{G}_{x,r} \simeq \underline{\mathfrak{g}}_{x,r}$, inflate to $G_{x,r}$ and extend by zero:

$$f_{x,r,\underline{\mathcal{O}}}(g) = \begin{cases} \Gamma_{\underline{\mathcal{O}}}(\overline{\log g}), & g \in G_{x,r}, \\ 0, & g \notin G_{x,r}. \end{cases}$$

The positive-depth tests satisfy the three properties

Ciubotaru–O. [CO25], Theorem 1.1

For $r \geq \text{depth}(\pi)$, the lifted graded Gelfand–Graev functions satisfy the three required properties for

$$\mathcal{O} = \mathcal{L}_{x,-r}(\underline{\mathcal{O}}).$$

For each rational nilpotent orbit $\mathcal{O}$, choose one pair $(x, \underline{\mathcal{O}})$ with $\mathcal{L}_{x,-r}(\underline{\mathcal{O}}) = \mathcal{O}$. Over these chosen pairs,

$$\text{WF}(\pi) = \max\{\mathcal{L}_{x,-r}(\underline{\mathcal{O}}) : \Theta_\pi(f_{x,r,\underline{\mathcal{O}}}) \neq 0\}.$$

$$\Theta_\pi(f_{x,r,\underline{\mathcal{O}}}) = \text{vol}(G_{x,r}) \langle \overline{\theta_{\pi,x,r}}, \Gamma_{\underline{\mathcal{O}}} \rangle.$$

Next question. When is $\langle \overline{\theta_{\pi,x,r}}, \Gamma_{\underline{\mathcal{O}}} \rangle$ nonzero?

How can we solve the rational intersection problem?

Assume G is unramified and $r \in \mathbb{Z}_{>0}$, $r \geq \text{depth}(\pi)$. Put $H = \overline{G}_x$ and $\mathfrak{h} = \text{Lie}(H)$, so $\mathfrak{g}_{x,\pm r} \simeq \mathfrak{h}(\mathbb{F}_q)$.

The orbit sum and character orthogonality give

$$\begin{aligned}\langle \overline{\theta_{\pi,x,r}}, \Gamma_e \rangle &= c_e \sum_{\mathcal{O}} m_{\pi,x,r}(\mathcal{O}) \sum_{X \in \mathcal{O}} \sum_{g \in H(\mathbb{F}_q)} \mathbf{1}_{\Sigma_e}(\text{Ad}(g)X) \\ &= c_e |H(\mathbb{F}_q)| \sum_{\mathcal{O}} m_{\pi,x,r}(\mathcal{O}) |\mathcal{O} \cap \Sigma_e(\mathbb{F}_q)|, \quad c_e > 0.\end{aligned}$$

The rational intersection problem

Given $X \in \mathfrak{h}(\mathbb{F}_q)$, which nilpotent e satisfy

$$H(\mathbb{F}_q) \cdot X \cap \Sigma_e(\mathbb{F}_q) \neq \emptyset?$$

Over $\overline{\mathbb{F}}_q$, this is an asymptotic-cone computation

The geometric asymptotic cone of the orbit of X is

$$\mathrm{AC}_{\overline{\mathbb{F}}_q}(H \cdot X) = \mathcal{N}(\mathfrak{h}) \cap \overline{\{t^2 \mathrm{Ad}(g)X : t \in \overline{\mathbb{F}}_q^\times, g \in H(\overline{\mathbb{F}}_q)\}}.$$

The bar denotes Zariski closure. Transversality and a contracting one-parameter subgroup of $\mathbb{G}_m \times H$ give the equivalence

$$e \in \mathrm{AC}_{\overline{\mathbb{F}}_q}(H \cdot X) \iff H(\overline{\mathbb{F}}_q) \cdot X \cap \Sigma_e(\overline{\mathbb{F}}_q) \neq \emptyset.$$

An asymptotic cone records directions at infinity

For an unbounded subset S of a real vector space, consider all limits

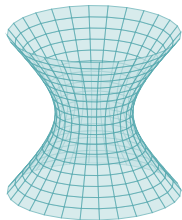
$$\text{AC}(S) = \{\lim_j t_j X_j : X_j \in S, t_j > 0, t_j \rightarrow 0\}.$$

We shrink the whole set towards the origin. Points that escape to infinity can leave nonzero limiting directions.

Real adjoint orbits in $\mathfrak{sl}_2$

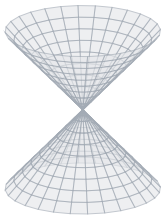
$$X = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}, \quad Q(X) = z^2 - x^2 - y^2.$$

Hyperbolic: $Q = -1$



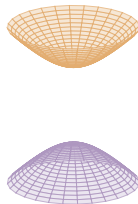
One orbit

Nilpotent: $Q = 0$



Two nonzero orbits
and the origin

Elliptic: $Q = 1$

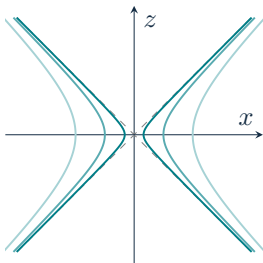


Each sheet is one orbit

Real asymptotic cones retain the orbit's rational form

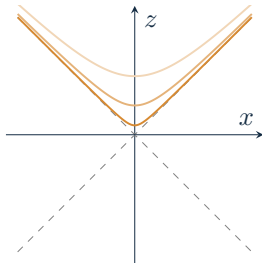
For $X = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$, $Q(X) = \det X = z^2 - x^2 - y^2$ and $Q(tX) = t^2 Q(X)$.

Hyperbolic: both halves survive



— $t = 1$ — $t = \frac{1}{2}$ — $t = .16$

Upper elliptic: upper half survives



— $t = 1$ — $t = \frac{1}{2}$ — $t = .16$

Cross-section $y = 0$: the upper elliptic orbit reaches only the upper nilpotent half-cone.

A rational slice distinguishes real orbit forms

Take $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = e^T$. Then

$$\mathcal{S}_e = e + \mathfrak{sl}_2^f = \left\{ \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

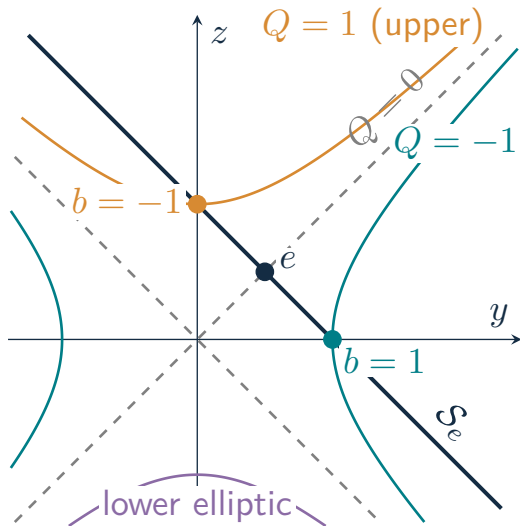
In the section $x = 0$, this is the line

$$y = \frac{1+b}{2}, \quad z = \frac{1-b}{2}, \quad Q = -b.$$

- $b > 0$: hyperbolic orbits.
- $b = 0$: the nilpotent element e .
- $b < 0$: upper elliptic orbits.

The lower elliptic sheet never meets the slice.

Rational asymptotic cones



The geometric cone and the rational question

For $X \in \mathfrak{h}(\mathbb{F}_q)$, we now return to the finite-field problem:

1. **Geometry over $\overline{\mathbb{F}}_q$.** Compute $\text{AC}_{\overline{\mathbb{F}}_q}(H \cdot X)$.
2. **Rational descent.** Determine which geometric degenerations are attained by slice intersections over $\mathbb{F}_q$.

The key rationality question is whether $\text{AC}_{\mathbb{F}_q}(H(\mathbb{F}_q) \cdot X)$ reaches the largest orbit in the geometric cone.

Recall: induction of nilpotent orbits

Over $\overline{\mathbb{F}}_q$, choose a parabolic $P = LU$ and write $\mathfrak{u} = \mathrm{Lie}(U)$.

Start with a nilpotent L -orbit $\mathcal{O}_L$ [LS79]. The induced orbit is the unique H -orbit meeting

$$\mathcal{O}_L + \mathfrak{u}$$

in a dense open subset.

The resulting orbit is independent of the parabolic with Levi L . This construction takes place over $\overline{\mathbb{F}}_q$.

A closed formula for the geometric cone

Take the Jordan decomposition and its semisimple centralizer:

$$X = X_s + X_n, \quad [X_s, X_n] = 0, \quad L = C_H(X_s).$$

Here L is a Levi subgroup, and $X_n \in \text{Lie}(L)$ is nilpotent.

Ciubotaru–O. [CO25], Theorem 1.6

$$\text{AC}_{\overline{\mathbb{F}}_q}(X) = \overline{\text{Ind}_L^H(L \cdot X_n)}.$$

Thus the cone is the closure of **one nilpotent orbit**. We denote that orbit by

$$\mathcal{N}(X) := \text{Ind}_L^H(L \cdot X_n).$$

Proof: sheets give the lower inclusion

Work over $\overline{\mathbb{F}}_q$. Recall that sheets are the irreducible components of

$$\mathfrak{h}^{(m)} = \{Y : \dim(H \cdot Y) = m\}.$$

On a sheet S , $\mathcal{N}(Y)$ is constant and is its unique nilpotent orbit. [B81] Choose $S \ni X$ and a triple (e, h, f) with $e \in \mathcal{N}(X)$. Every orbit in S meets the Slodowy slice $e + \mathfrak{h}^f$. Thus some $Y \in H \cdot X$ lies in that slice, and

$$t^2 \operatorname{Ad}(\lambda(t)^{-1})Y \longrightarrow e \quad (t \rightarrow 0).$$

Since the asymptotic cone is closed and H -stable,

$$\overline{\mathcal{N}(X)} \subseteq \operatorname{AC}_{\overline{\mathbb{F}}_q}(X).$$

The reverse inclusion follows from standard closure results of Borho. [B81]

The rationality question

For $X \in \mathfrak{h}(\mathbb{F}_q)$, geometry gives

$$\mathrm{AC}_{\overline{\mathbb{F}}_q}(X) = \overline{\mathcal{N}(X)}.$$

Does the rational cone reach the largest orbit?

$$\mathrm{AC}_{\mathbb{F}_q}(X) \cap \mathcal{N}(X)(\mathbb{F}_q) \stackrel{?}{\neq} \emptyset.$$

In this ungraded setting, an orbit meets the weak slice if and only if it meets the Slodowy slice, also over $\mathbb{F}_q$. Thus we seek a rational triple whose Slodowy slice contains X . Define

$$\Theta(X) = \{\phi = (e, h, f) \text{ an } \mathfrak{sl}_2\text{-triple} : e \in \mathcal{N}(X), X \in e + \mathfrak{h}^f\}.$$

We know $\Theta(X)$ nonempty over $\overline{\mathbb{F}}_q$; rationality asks for a Frobenius-fixed point.

Good orbits and connected transitivity

Put $H_X = C_H(X)$ and $H_\phi = C_H(e, h, f)$ for $\phi = (e, h, f) \in \Theta(X)$. Katsylo's theorem [Kat83] gives $\Theta(X) = H_X \cdot \phi$.

Call X *good* if the connected centralizer is already transitive:

$$\Theta(X) = H_X^\circ \cdot \phi \iff H_X^\circ H_\phi = H_X H_\phi.$$

Semisimple and nilpotent elements are good.

Tsai [T23], Theorem B.4

Under the standing hypotheses, every $X \in \mathfrak{h}$ is good.

Lang's theorem [Lan56]: every nonempty homogeneous space of a connected algebraic group over $\mathbb{F}_q$ has an $\mathbb{F}_q$ -point. Thus, for $X \in \mathfrak{h}(\mathbb{F}_q)$,

$$\Theta(X)(\mathbb{F}_q) \neq \emptyset.$$

The largest orbit is reached over the finite field

Now suppose $X \in \mathfrak{h}(\mathbb{F}_q)$, and let $\mathcal{O}_X = H(\mathbb{F}_q) \cdot X$.

Rational attainment: [CO25], Theorem 1.6

$$\boxed{\mathrm{AC}_{\mathbb{F}_q}(\mathcal{O}_X) \cap \mathcal{N}(X)(\mathbb{F}_q) \neq \emptyset.}$$

More concretely, there is a $\mathbb{F}_q$ -rational triple (e, h, f) such that

$$e \in \mathcal{N}(X)(\mathbb{F}_q), \quad X \in e + \mathfrak{h}^f.$$

Thus the rational cone reaches the largest geometric orbit, but it need not contain every rational point of the geometric cone.

One point and one Fourier orbit give a singleton

Suppose $r = \text{depth}(\pi) \in \mathbb{Z}_{>0}$. In an orbit-detecting set $\mathcal{C}$, assume only one point x contributes:

$$V^{G_{y,r^+}} = 0 \quad (y \in \mathcal{C} \setminus \{x\}), \quad \text{supp } \hat{\theta}_{\pi,x,r} = \overline{G}_x(\mathbb{F}_q) \cdot \xi.$$

The geometric cone is $\overline{\mathcal{N}(\xi)}$, and rational attainment reaches its largest orbit. Hence $\text{WF}_{\text{geom}}(\pi)$ is a singleton, obtained by lifting

$$\mathcal{N}(\xi) = \text{Ind}_{C_{\overline{G}_x}(\xi_s)}^{\overline{G}_x}(C_{\overline{G}_x}(\xi_s) \cdot \xi_n).$$

This occurs for a substantial family of toral supercuspidal representations.

Toral supercuspidals: starting from a torus

A toral supercuspidal is constructed from a generic positive-depth character of an elliptic maximal torus, using a single positive-depth step of Yu's construction and compact induction.

Here we use the unramified DeBacker–Reeder family. [DR10] Let G be semisimple and unramified, and choose a regular good element

$$X \in \mathfrak{g} \text{ of depth } -r, \quad S = C_G(X) \text{ an unramified elliptic torus.}$$

Regular goodness gives the depth data

Recall that X has depth $-r$ and $S = C_G(X)$ is unramified elliptic. Regular good means that X is regular semisimple and

$$\mathrm{val}(d\alpha(X)) = -r \quad \text{for every root } \alpha \in \Phi(G, S).$$

Via B and the additive character, X specifies the depth- r character; write Π_X for the resulting family.

For $\pi \in \Pi_X$, r is integral. The unique fixed point x of S supplies the single contributing point and single Fourier orbit described earlier.

Reeder–Yu: the reductive quotient as a fixed subgroup

Work over $\overline{\mathbb{F}}_q$. Put $H = \overline{G}_x$, and let H_0 be the reductive quotient at a hyperspecial point over F^{un} . Thus H_0 has the same absolute root datum as G .

Reeder–Yu realise the Moy–Prasad quotients as eigenspaces of a finite-order automorphism ϑ_x of $\mathfrak{h}_0 = \text{Lie}(H_0)$. Since G is unramified, ϑ_x is inner.

Reeder–Yu, Theorem 4.1

For a suitable finite-order semisimple element $s \in H_{0,\text{ad}}$,

$$H \simeq (H_0^{\vartheta_x})^\circ, \quad \text{Lie}(H) \simeq \mathfrak{h}_0^{\vartheta_x} \subset \mathfrak{h}_0, \quad \vartheta_x = \text{Ad}(s).$$

This supplies the embedding in which we compare nilpotent orbits. [Reeder–Yu, 2014]

Which root spaces survive in the reductive quotient?

Choose apartment coordinates with a hyperspecial origin, and a maximal torus $T_0 \subset H_0$.
Write

$$\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{h}_{0,\alpha}.$$

Choose m with $m\alpha(x) \in \mathbb{Z}$ for every root and a primitive m th root of unity ζ . The grading automorphism acts by

$$\vartheta_x|_{\mathfrak{h}_{0,\alpha}} = \zeta^{m\alpha(x)}, \quad \vartheta_x|_{\mathfrak{t}_0} = 1.$$

Consequently,

$$\mathrm{Lie}(H) \simeq \mathfrak{h}_0^{\vartheta_x} = \mathfrak{t}_0 \oplus \bigoplus_{\alpha(x) \in \mathbb{Z}} \mathfrak{h}_{0,\alpha}.$$

These are precisely the roots of the reductive quotient at x .

A nilpotent element regular in this subalgebra need not be regular in $\mathfrak{h}_0$.

Computing the geometric wavefront orbit of the toral family

Since r is integral, view the reduction $\underline{X}$ in $\mathrm{Lie}(H)$. Regular goodness makes it regular semisimple. Hence

$$\mathcal{N}(\underline{X}) = \mathcal{O}_{\mathrm{reg}}^H.$$

Choose $e \in \mathcal{O}_{\mathrm{reg}}^H$ and use the Reeder–Yu embedding:

$$e \in \mathrm{Lie}(H) \hookrightarrow \mathrm{Lie}(H_0) \quad \longmapsto \quad H_0 \cdot e.$$

Weighted Dynkin diagrams identify geometric nilpotent orbits for H_0 and G across the two characteristics, and lifting preserves weighted Dynkin diagrams.

Therefore π is generic iff $H_0 e$ is generic iff $C_{H_0, \mathrm{ad}}(e)_{\mathrm{red}} = 1$ iff x is hyperspecial.

This recovers DeBacker–Reeder’s genericity criterion for toral supercuspidal representations.

The degeneration problem at rational depth

At nonintegral rational depth, we return to a cyclically graded Lie algebra

$$\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}/n} \mathfrak{l}_i, \quad \mathfrak{l}_{-1} \simeq \underline{\mathfrak{g}}_{x,-r}, \quad H = \overline{G}_x.$$

Which nilpotent orbits are degenerations of X ?

Given $X \in \underline{\mathfrak{g}}_{x,-r}(\mathbb{F}_q)$, determine the nilpotent $H(\mathbb{F}_q)$ -orbits of e for which

$$H(\mathbb{F}_q) \cdot X \cap \Sigma_e(\mathbb{F}_q) \neq \emptyset.$$

We still do not know how to compute these rational intersections in general.

Also rational attainment is an issue: in Tsai's unitary epipelagic example,

$$H_X^\circ = \{1\} \quad \text{does not act transitively on } \Theta_r(X).$$

**Towards an upper bound
for full singular supports**

The character side

Left translation identifies

$$\mathrm{SS}(\Theta_\pi) \subset T^*G \simeq G \times \mathfrak{g}^*,$$

Let $p : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ be the projection.

Restrict to the identity fibre:

$$\mathrm{SS}_1(\Theta_\pi) := \mathrm{SS}(\Theta_\pi) \cap (\{1\} \times \mathfrak{g}^*).$$

Its projected maximal nilpotent orbits are precisely the wavefront set:

$$\boxed{\mathrm{WF}(\pi) = \max\{\mathcal{O} : \mathcal{O} \subset p(\mathrm{SS}_1(\Theta_\pi))\}.$$

The parameter side

Fix an infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ and put

$$V_\lambda = \{N \in \mathfrak{g}^\vee : \mathrm{Ad}(\lambda(w))N = |w|N \text{ for all } w \in W_F\},$$

$$H_\lambda = Z_{G^\vee}(\lambda).$$

The enhanced LLC attaches to π an H_λ -orbit $\mathcal{O}_\phi$ and a local system $\mathcal{L}_\rho$, hence the perverse sheaf

$$\mathcal{F}_\pi = \mathrm{IC}(\overline{\mathcal{O}_\phi}, \mathcal{L}_\rho).$$

Its singular support lies in

$$\mathrm{SS}(\mathcal{F}_\pi) \subset T^*V_\lambda = V_\lambda \times V_\lambda^*.$$

Aubert duality and projected singular support

Let π be an irreducible unipotent representation of an inner form of an adjoint unramified group, and $D\pi$ its irreducible Aubert dual.

Consequence of Antor–O., Theorem A [AO26]

After $G^\vee$ -saturation and Zariski closure on both sides,

$$\mathrm{supp}(\mathcal{F}_{D\pi}) = p^\vee(\mathrm{SS}(\mathcal{F}_\pi)).$$

The upper bound in terms of singular supports

The parameter support in the upper bound can therefore be replaced by the projection of singular support.

We can therefore consider the following diagram:

$$\begin{array}{ccc} \mathrm{SS}(\Theta_\pi) & \xrightarrow{p} & \mathcal{N}(\mathfrak{g}^*) \\ & & \uparrow d^\vee \\ \mathrm{SS}(\mathcal{F}_\pi) & \xrightarrow{p^\vee} & \mathcal{N}((\mathfrak{g}^\vee)^*) \end{array}$$

Reformulation of the upper bound conjecture

$$\begin{aligned} p(\mathrm{SS}_1(\Theta_\pi)) \\ \subseteq d^\vee(p^\vee(\mathrm{SS}(\mathcal{F}_\pi))) . \end{aligned}$$

What happens if we remove the 1?

Restriction to the identity gives an automatic inclusion:

$$p(\mathrm{SS}_1(\Theta_\pi)) \subseteq p(\mathrm{SS}(\Theta_\pi)).$$

Theorem/Conjecture (folklore)

$$\boxed{p(\mathrm{SS}(\Theta_\pi)) = p(\mathrm{SS}_1(\Theta_\pi)).}$$

Proved for unitary admissible representations of semisimple p -adic groups over a characteristic-zero local field by Heifetz, Theorem 3.4. [Hei85].

A geometric realization?

Can the upper bound conjecture be realised geometrically through these two singular supports?

$$\begin{array}{ccc} \mathrm{SS}(\Theta_\pi) & & \mathrm{SS}(\mathcal{F}_\pi) \\ \downarrow p & & \downarrow p^\vee \\ \mathcal{N}(\mathfrak{g}^*) & \xleftarrow{d^\vee} & \mathcal{N}((\mathfrak{g}^\vee)^*) \end{array}$$

**An Atlas for Arthur packets of exceptional groups,
and
A characterisation of microlocal Arthur packets as a
quasi-canonical basis of stable distributions**

Arthur Atlas: the exceptional packets are now explicit

Arxiv paper: <https://arxiv.org/pdf/2610.16931>
Atlas: emileokada.com/arthur-atlas/

All unipotent ABV Arthur packets with unramified real infinitesimal character for exceptional adjoint groups inner to split:

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8.$$

- Explicit Arthur packet membership tables.
- Orbit and local-system parameters, IC polynomials and Fourier duality.



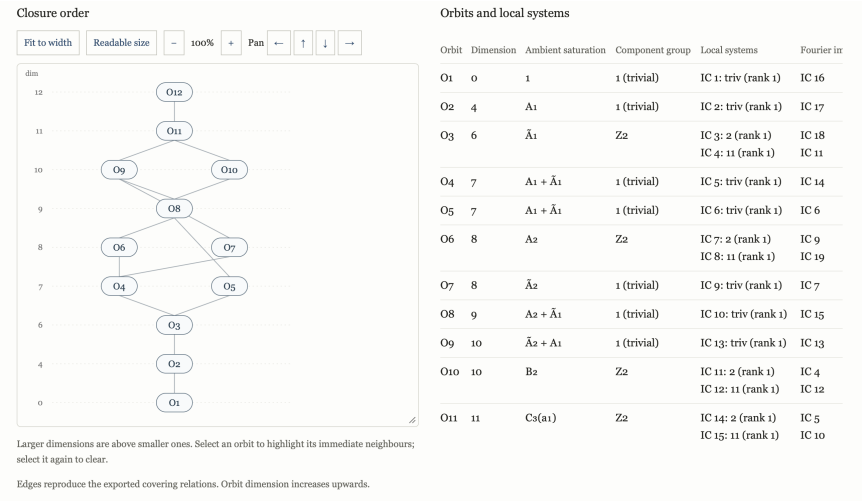
The computational scale of E_8

The complete dataset covers all 70 nilpotent-orbit cases (609,529 polynomial-matrix entries).

Final 12-orbits on a Mac mini M4 Pro

Hardware	14-core CPU; 48 GB unified memory
Execution	1 worker, 2 compute threads
Elapsed time / peak resident memory	68 hours / 19.36 GiB
Largest chart $W(E_8)/W(h)$	$696,729,600/2 = 348,364,800$ cosets
Cumulative scratch-cache volume	≈ 684.5 GiB; 250 GiB scratch budget

Arthur Atlas: orbits and local systems

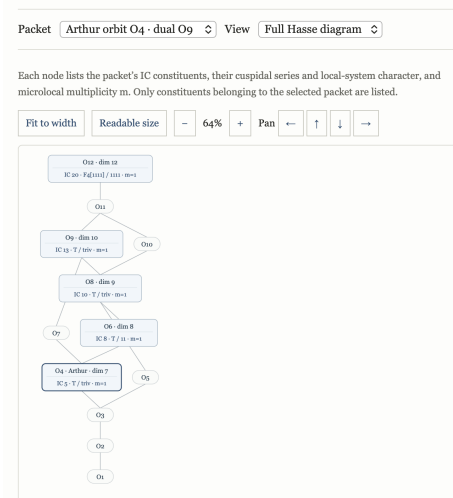


Arthur Atlas: packet members and multiplicities

Arthur packets

Arthur orbit	Dual orbit	Members (IC : multiplicity)
O1	O12	1 : 1, 2 : 3, 3 : 2, 8 : 3, 20 : 1
O2	O11	2 : 1, 3 : 1, 5 : 1, 8 : 2, 10 : 1, 20 : 1
O3	O10	3 : 1, 4 : 1, 8 : 1, 10 : 2, 12 : 1, 20 : 1
O4	O9	5 : 1, 8 : 1, 10 : 1, 13 : 1, 20 : 1
O6	O7	7 : 1, 8 : 1, 13 : 2, 20 : 1
O7	O6	9 : 1, 13 : 2, 19 : 1, 20 : 1
O8	O8	10 : 1, 12 : 1, 13 : 1, 15 : 1, 20 : 1
O9	O4	13 : 1, 14 : 1, 15 : 1, 19 : 1, 20 : 1
O10	O3	11 : 1, 12 : 1, 15 : 2, 18 : 1, 19 : 1, 20 : 1
O11	O2	14 : 1, 15 : 1, 17 : 1, 18 : 1, 19 : 2, 20 : 1
O12	O1	16 : 1, 17 : 3, 18 : 2, 19 : 3, 20 : 1

Arthur Atlas: explore an individual packet



Arthur packets from characteristic cycles

New joint work with Dan Ciubotaru: *Microlocal Arthur packets as a quasi-canonical basis of stable distributions*. [CO26]

Fix an infinitesimal parameter λ . For an irreducible π , write

$$\mathrm{CC}(\mathcal{F}_\pi) = \sum_{C \in H_\lambda \setminus V_\lambda} m_C(\mathcal{F}_\pi) [\overline{T_C^* V_\lambda}].$$

An Arthur parameter ψ determines an orbit $C_\psi \subset V_\lambda$. Its microlocal (ABV) packet is

$$\Pi_\psi^{\mathrm{mic}} = \{\pi : m_{C_\psi}(\mathcal{F}_\pi) > 0\}.$$

Arthur distributions as a quasi-canonical basis

For an L-parameter ϕ with Langlands classification data $(M^\vee, \phi_{M^\vee}, z)$ let

$$S_\phi = \sum_G i_{P,M}^G(\eta_{\phi_M}^G \otimes \chi_z^G).$$

Let $\mathcal{F}$ denote the transported Fourier involution and S_ϕ the standard Langlands packet sums across pure inner forms. Set

$$U_{>\psi} = \text{span}_{\mathbb{C}}\{S_\phi : C_\phi > C_{\phi_\psi}\}.$$

Arthur distributions as a quasi-canonical basis

Characterization (Ciubotaru–O., Theorem 1.2)

The microlocal Arthur distributions form the unique family satisfying:

1. Triangularity:

$$T_\psi \equiv S_{\phi_\psi} \pmod{U_{>\psi}}.$$

2. Compatibility with Fourier duality:

$$\mathcal{F}(T_\psi) = \epsilon_\psi T_{\psi^t}, \quad \epsilon_C = (-1)^{\dim C_\psi - \dim C_{\psi^t}}.$$

$$\left(\mathcal{F}(T_\psi) \equiv \epsilon_\psi S_{\phi_{\psi^t}} \pmod{U_{>\psi^t}} \right).$$

A characterization using the Aubert involution

Assume we have an enhanced tempered LLC of G and its Levis, the p -adic Kazhdan–Lusztig conjectures and Aubert compatibility:

- for every Arthur orbit A and irreducible π ,

$$A \leq C(|AZ(\pi)|) \iff A \leq C(\mathcal{F}\mathcal{F}_\pi).$$

(proved by '26 Antor-O. for unipotent representations of adjoint unramified groups).

Then T_ψ is the unique stable distribution satisfying

$$T_\psi \equiv S_{\phi_\psi} \pmod{U_{>\psi}}, \quad AZ(T_\psi) = \pm T_{\psi^t}.$$

Consequences for Arthur packets

- **Agreement with endoscopic packets.** For split $\mathrm{SO}_{2n+1}(F)$ in Lusztig's unipotent category,

$$\Pi_{\psi}^{\mathrm{mic}}(G(F)) = \Pi_{\psi}^{\mathrm{Art}}(G(F))$$

for every unramified ψ , when $\mathrm{char} F = 0$ and the residue characteristic is at least $6n + 4$.

- **Weak Arthur packets.** The weak Arthur packets conjecture holds for exceptional groups.
- **Extremality can fail.** A fixed representation can belong to packets with two minimal Arthur parameters ($F_4(a_3)$), or two maximal ones ($E_8(a_7)$).

The key: rigidity of Arthur orbits

Write $C \leq D$ for $C \subseteq \overline{D}$, and $C^\dagger$ for the Piasetskii-dual orbit (identifying the dual parameter space with V_λ). For Arthur parameters, $C_\psi^\dagger = C_{\psi^t}$, where ψ^t exchanges the two SL_2 factors.

Rigidity (Ciubotaru–O., Theorem 1.1)

If C is Arthur and D is any orbit, then

$$C \leq D, \quad C^\dagger \leq D^\dagger \implies C = D.$$

Kashiwara's index formula gives linear equations for the multiplicities. Fourier transform supplies the symmetry

$$m_{C^\dagger}(\mathcal{F}\mathcal{F}) = m_C(\mathcal{F}).$$

Together with rigidity, these determine every Arthur row uniquely.

Thank you!

Proof: the central family gives the upper inclusion

Put $\mathfrak{l} = \text{Lie}(L)$ and $\mathfrak{z} = Z(\mathfrak{l})$, where $L = C_H(X_s)$. Since $X_s \in \mathfrak{z}$ and $L \cdot X_n$ is stable under nonzero scalars,

$$\{t^2 \text{Ad}(g)X : t \neq 0, g \in H\} \subseteq H \cdot (\mathfrak{z} + L \cdot X_n).$$

Borho's closure result [B81], Lemma 2.5(b), gives

$$\mathcal{N}(\mathfrak{h}) \cap \overline{H \cdot (\mathfrak{z} + L \cdot X_n)} \subseteq \overline{\text{Ind}_L^H(L \cdot X_n)}.$$

Intersecting the first inclusion's closure with the nilpotent cone yields

$$\text{AC}_{\overline{\mathbb{F}_q}}(X) \subseteq \overline{\mathcal{N}(X)}.$$

Together, the two inclusions prove the formula.

An example: a semisimple element in $\mathfrak{gl}_3$

Let $X = \text{diag}(a, a, b)$ with $a \neq b$. Then

$$X_n = 0, \quad L = \text{GL}_2 \times \text{GL}_1.$$

The off-diagonal matrices in the corresponding nilradical have the form

$$\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}.$$

For $(u, v) \neq (0, 0)$, this matrix has rank one and square zero, hence Jordan type $(2, 1)$.

$$\boxed{\mathcal{N}(X) = \mathcal{O}_{(2,1)}, \quad \text{AC}_{\overline{\mathbb{F}}_q}(X) = \overline{\mathcal{O}_{(2,1)}}.}$$

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